

Chern-Simons theory and BCS superconductivity

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Abstract

We study the relationship between the holomorphic unitary connection of Chern-Simons theory with temporal Wilson lines and the Richardson's exact solution of the reduced BCS Hamiltonian. We derive the integrals of motion of the BCS model, their eigenvalues and eigenvectors as a limiting case of the Chern-Simons theory.

1 Introduction

Chern-Simons theory has emerged in the last years as a very useful field theoretical model for the description of interesting phenomena in Condensed Matter Physics. It plays a leading role in the description of the quantum Hall effect [1]. It has been also advocated to be connected with high T_c superconductivity [2]. In this paper we will show that it also appears to be closely related to the Bardeen, Cooper and Schrieffer (BCS) theory of superconductivity [3].

One of the corner stones in Condensed Matter Physics is the pairing model of Bardeen, Cooper and Schrieffer (BCS) which explains the properties of “low T_c ” superconductors [3] and several nuclei [4]. The BCS model was solved in the Grand Canonical Ensemble by a variational ansatz, which is asymptotically exact in the number of fermions forming pairs. A reduced version of the BCS model, where the pairing interaction has the same strength between all pairs, was solved exactly in 1963 by Richardson in the Canonical Ensemble, where the number of Cooper pairs is fixed [5, 6, 7].

Despite of the interest of this exact solution it escaped the attention of the condensed matter and nuclear physics communities, until recently with the advent of ultra-small

superconducting grains [8], whose study is based on the BCS model with a fixed number of pairs (for a review see [9]).

Partially motivated by Richardson’s work, in 1976 Gaudin proposed a family of commuting spin Hamiltonians whose exact diagonalization paralleled Richardson’s solution [10]. However a relation between the two models was not clear at that time, despite of the fact that in the limit where the BCS coupling constant g goes to infinity, the Richardson’s solution turns into the Gaudin’s one.

A first step to understand the relation between the Gaudin’s spin Hamiltonians and the reduced BCS Hamiltonian came with the work of Cambiaggio, Rivas and Saraceno (CRS) who proved the integrability of the reduced BCS model in terms of a family of commuting operators, which are nothing but the Gaudin’s Hamiltonians plus a perturbation proportional to the inverse of the BCS coupling constant g [11]. CRS also constructed the BCS Hamiltonian as a linear combination of the perturbed Gaudin’s operators. Unfortunately these authors were unaware of Richardson’s and Gaudin’s works and consequently they did not obtained the eigenvalues of the BCS conserved quantities.

The eigenvalues of the CRS operators were given in the reference [12], using Conformal Field Theory (CFT) methods, and generalized to other models in [13, 14, 15]. The aim of reference [12] was to place the Richardson’s exact solution and its integrability in the framework of CFT, relating it to the Wess-Zumino-Witten models with Kac-Moody algebra $SU(2)_k$. Using CFT methods it was shown that the Richardson’s wave function are obtained as conformal blocks of a perturbed $SU(2)_k$ -WZW model when the level k goes to a critical value -2 [12].

The CFT interpretation of the BCS model turned out to be closely related to the work of Babujian [16], who in 1993, used the so called off-shell algebraic Bethe ansatz, to re-derived Gaudin’s exact solution [16, 17]. Babujian also observed that the Gaudin’s eigenstates can be used to build the conformal blocks of the WZW models.

The similarity between the works [12] and [16, 17] suggested that the Richardson’s solution of the reduced BCS model should also be derivable using the off-shell Bethe ansatz method. This was done by Amico, Falci and Fazio [18] and later on clarified in references [19, 20, 21], where the BCS coupling constant parametrizes a boundary operator that appears in the transfer matrix of the inhomogenous vertex model, whose semi-classical limit gives rise to the CRS conserved quantities.

The old results by Richardson and Gaudin’s, concerning the norm of the eigenstates and the occupation numbers [22, 23], has been generalized to other operators in references [24, 19] using the “determinantal” techniques developed in [25, 26].

The outcome of all these works has been to clarify the integrability and relationship between the reduced BCS and Gaudin’s model at least at a formal level. There remains however the question concerning the “geometrical” or field theoretical origin of the integrability of BCS. For the Gaudin’s Hamiltonians this is given by the underlying WZW model, and ultimately by the Chern-Simons theory [27]. The BCS model is related to some sort of chiral perturbation of the WZW model, characterized by modified Kniznik-Zamolodchikov equations [12, 19, 21].

In this paper we shall show that the field theoretical origin of BCS can be traced back to a $SU(2)$ Chern-Simons (CS) theory interaction with a one-dimensional distribution

of coloured matter which breaks both gauge and conformal invariance. This connection is quite remarkable because Chern-Simons theory has been advocated to be mainly connected with effective descriptions of fractional quantum Hall effect and high T_c superconductivity, but never with standard BCS superconductivity. The Chern-Simons theory is not defined in the physical space, which might be two or three-dimensional, but rather in the complex energy plane which is always two-dimensional. This explains why this field theoretical connection of BCS theory remained unveiled for so long time. The connection of Chern-Simons theory with BCS model can be understood in a more general framework when we consider a scaling limit of the twisted Chern-Simons theory defined on a torus, that is, the twisted elliptic Chern-Simons theory. On a torus the KZ equations [28] are replaced by the Knizhnik-Zamolodchikov-Bernard (KZB) equations [29], which depend on the coordinates z_n of the punctures, the moduli of the torus τ and a set of parameters u_j characterizing the toroidal flat gauge connections. The later parameters u_j define the twisted boundary conditions for the WZW fields on the torus.

The main result of this paper is to show that, for a generic simple simply connected, compact Lie group G , the Richardson equations, the CRS conserved quantities and their eigenvalues arise from the KZB connection and their associated horizontal sections. This will be done in a limit where the torus degenerates into the cylinder and then into the complex plane. In this limiting procedure the generalized BCS coupling constants appear as conjugate variables of the parameters u_j , when this parameters go to infinity. This gives the G -based BCS models a suggestive geometrical and group theoretical meaning.

The organization of the paper is as follows. In section 2 we briefly review the reduced BCS model and its exact solution. In section 3 we consider a perturbation of the CS model defined on the plane, we derive the KZ equations and study their connection with the CRS conserved quantities and the BCS model. In section 4 we introduce the KZB connection related to the CS theory on the torus. In section 5 we derive the Bethe ansatz of the integrable Hamiltonians introduced in sections 3 and 4. Finally we state our conclusions. We have also included an appendix with particular examples of the general equations contained in the main text.

2 Review of the exact solution of the BCS model

The reduced BCS model is defined by the Hamiltonian [3, 9, 12]

$$H_{BCS} = \sum_{n,\sigma=\pm} \varepsilon_{n\sigma} c_{n\sigma}^\dagger c_{n\sigma} - gd \sum_{n,n'} c_{n+}^\dagger c_{n-}^\dagger c_{n'-} c_{n'+} \quad (1)$$

where $c_{n,\pm}$ (resp. $c_{n,\pm}^\dagger$) is an electron destruction (resp. creation) operator in the time-reversed states $|n,\pm\rangle$ with energies ε_n , d is the mean level spacing and g is the BCS dimensionless coupling constant. The sums in (1) run over a set of N doubly degenerate energy levels $\varepsilon_n (n = 1, \dots, N)$. We shall assume that the energy levels are all distinct, i.e. $\varepsilon_m \neq \varepsilon_n$ for $m \neq n$. The Hamiltonian (1) is a simplified version of the reduced BCS Hamiltonian where all couplings have been set equal to a single one, namely g . This is the model that is commonly used to describe ultrasmall grains and accounts for the

scattering of pairs of electrons between discrete energy levels that come in time-reversed states. Hereafter we shall refer to (1) simply as the BCS Hamiltonian.

Richardson had long ago solved this model exactly for an arbitrary set of levels, ε_n [5, 6, 7]. To simplify matters, we shall assume that there are not singly occupied electronic levels. As can be seen from (1), these levels decouple from the rest of the system; they are said to be blocked, contributing only with their energy ε_n to the total energy \mathcal{E} . The above simplification implies that every energy level n is either empty (i.e. $|\text{vac}\rangle$), or occupied by a pair of electrons (i.e. $c_{n,+}^\dagger c_{n,-}^\dagger |\text{vac}\rangle$). Denote the total number of electrons pairs by M . Then of course $M \leq N$. The most studied case in the literature corresponds to the half-filled situation, where the number of electrons, $N_e = 2M$, is equal to the number of levels N [9]. In the absence of interaction (i.e. $g = 0$), all the pairs occupy the lowest energy levels forming a Fermi sea. The pairing interaction promotes the pairs to higher energies and eventually, for large values of M , all the levels are pair correlated, giving rise to superconductivity [3].

In order to describe Richardson's solution one defines the hard-core boson operators

$$b_n = c_{n,-} c_{n,+}, \quad b_n^\dagger = c_{n,+}^\dagger c_{n,-}^\dagger, \quad N_n = b_n^\dagger b_n \quad (2)$$

which satisfy the commutation relations,

$$[b_n, b_{n'}^\dagger] = \delta_{n,n'} (1 - 2N_n) \quad (3)$$

The Hamiltonian (1) can then be written as

$$H_{BCS} = \sum_n 2\varepsilon_n b_n^\dagger b_n - g \sum_{n,n'} b_n^\dagger b_{n'}, \quad (4)$$

where we have set $d = 1$ (i.e. all the energies are measured in units of d). Richardson showed that the eigenstates of this Hamiltonian with M pairs have the (unnormalized) product form [5, 6, 7]

$$|M\rangle_R = \prod_{j=1}^M B_j |\text{vac}\rangle, \quad B_j = \sum_{n=1}^N \frac{1}{2\varepsilon_n - E_j} b_n^\dagger \quad (5)$$

where the parameters E_j ($j = 1, \dots, M$) are, in general, complex solutions of the M coupled algebraic equations

$$\frac{1}{g} + \sum_{j'=1(\neq j)}^M \frac{2}{E_{j'} - E_j} = \sum_{n=1}^N \frac{1}{2\varepsilon_n - E_j}, \quad (6)$$

which are a sort of Bethe ansatz equations for this problem. The energy of these states is given by the sum of the auxiliary parameters E_j , i.e.

$$\mathcal{E}(M) = \sum_{j=1}^M E_j \quad (7)$$

The ground state of H_{BCS} is given by the solution of eqs.(6) which gives the lowest value of $\mathcal{E}(M)$. The (normalized) states (5) can also be written as [22]

$$|M\rangle_R = \frac{C}{\sqrt{M!}} \sum_{n_1, \dots, n_M=1}^N \psi^R(n_1, \dots, n_M) b_{n_1}^\dagger \cdots b_{n_M}^\dagger |\text{vac}\rangle \quad (8)$$

where the sum excludes double occupancy of pair states ($n_i \neq n_j$ for $i \neq j$) and the wave function ψ takes the form

$$\psi^R(n_1, \dots, n_M) = \sum_{\mathcal{P} \in S_M} \prod_{k=1}^M \frac{1}{2\varepsilon_{n_k} - E_{\mathcal{P}k}} \quad (9)$$

The sum in (9) runs over the set of permutations S_M , of $1, \dots, M$. The constant C in (8) guarantees the normalization of the state [22] (i.e. ${}_R\langle N|N\rangle_R = 1$).

A well known fact about the BCS Hamiltonian is that it is equivalent to that of a XY model with long range couplings and a “position dependent” magnetic field proportional to ε_n . To see this let us represent the hard-core boson operators (2) in terms of the Pauli matrices as follows,

$$b_n = \sigma_n^+, b_n^\dagger = \sigma_n^-, N_n = \frac{1}{2}(1 - \sigma_z) \quad (10)$$

in which case the Hamiltonian (4) becomes

$$\begin{aligned} H_{BCS} &= H_{XY} + \sum_{n=1}^N \varepsilon_n + g(N/2 - M) \\ H_{XY} &= - \sum_{n=1}^N 2\varepsilon_n \tau_n^0 - \frac{g}{2}(T^+ T^- + T^- T^+) \end{aligned} \quad (11)$$

where the matrices

$$T^a = \sum_{n=1}^N \tau_n^a \quad (a = 0, +, -) \quad (12)$$

$$\tau_n^0 = \frac{1}{2}\sigma_n^z, \quad \tau_n^+ = \sigma_n^+, \quad \tau_n^- = \sigma_n^-$$

satisfy the $SU(2)$ algebra,

$$\begin{aligned} [T^a, T^b] &= f_c^{ab} T^c \\ f_+^{+0} &= f_-^{0-} = -1, \quad f_0^{+-} = 2 \end{aligned} \quad (13)$$

whose Casimir is given by

$$\mathbf{T} \cdot \mathbf{T} = T^0 T^0 + \frac{1}{2}(T^+ T^- + T^- T^+) \quad (14)$$

This spin representation of the BCS model is the appropriate one to study its integrability, as was shown by Cambiaggio, Rivas and Saraceno by explicitly constructing a set of operators, R_n ($n = 1, \dots, N$), which commute with H_{BCS} and moreover allow a reconstruction of the BCS Hamiltonian which is given by $H_{BCS} = g \sum_n 2\epsilon_n R_n + \text{ctes.}$ In the next sections we shall generalize this construction to arbitrary Lie groups and representations.

3 Chern-Simons Theory and BCS Hamiltonian.

Let G be a simple, simply connected, compact Lie group. The Chern-Simons theory of N interacting heavy coloured particles (temporal Wilson line insertions) located at points z_1, z_2, \dots, z_N of the 2D plane \mathbb{C} is defined by the action [30]

$$S(A, g_1, \dots, g_N) = \frac{k}{4\pi} \int \text{tr} \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \sum_{n=1}^N \int_{-\infty}^{\infty} dt \langle \Lambda_n, g_n^{-1}(t) A_0(z_n, t) g_n(t) + g_n^{-1}(t) \frac{d}{dt} g_n(t) \rangle \quad (15)$$

where A is a G -gauge field, g_n a G -valued chiral field describing the dynamics of the coloured particle located at the insertion point z_n and Λ_n the highest weight of its G -colour representation. $\langle \cdot, \cdot \rangle$ denotes the Killing scalar product in the Lie algebra \mathfrak{g} of G normalized so that long roots have length 2, we use this scalar product to identify \mathfrak{g} and its dual \mathfrak{g}^* . The quantization of the theory (15) yields quantum states which have to satisfy two constraints. In the Bargmann representation they are holomorphic functionals $\Psi(A_{\bar{z}})$ of the $(0, 1)$ component of the gauge field $A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2)$ and take values in the tensor product of the representation spaces $\otimes V_{\Lambda_n}$ of the insertions. The remaining constraint is Gauss' law

$$D_{\bar{z}}^{ab} \frac{\delta}{\delta A_{\bar{z}}^b(z)} \Psi(A_{\bar{z}}) = -\frac{k}{\pi} \partial_z A_{\bar{z}}^a \Psi(A_{\bar{z}}) + \sum_{n=1}^N t_{(n)}^a \delta(z - z_n) \Psi(A_{\bar{z}}), \quad (16)$$

where t^a is an orthonormal basis of the Lie algebra \mathfrak{g} of G and $t_{(n)}^a$ denotes the operator representation on V_{Λ_n} of $t^a \in \mathfrak{g}$. The law (16) governs the dependence of the quantum states Ψ under a $G^{\mathbb{C}}$ -valued complex gauge transformation g , i.e.

$$\Psi(g^{-1} A_{\bar{z}} g + g^{-1} \partial_{\bar{z}} g) = e^{k S_{WZW}(A_{\bar{z}}, g)} \prod_{n=1}^N g(z_n)_{(n)}^{-1} \Psi(A_{\bar{z}}) \quad (17)$$

where

$$S_{WZW}(A_{\bar{z}}, g) = -\frac{i}{4\pi} \int dz d\bar{z} \text{tr} g^{-1} (\partial_z g) g^{-1} \partial_{\bar{z}} g - \frac{i}{12\pi} \int d^{-1} \text{tr} (g^{-1} dg)^3 + \frac{i}{2\pi} \int dz d\bar{z} \text{tr} A_{\bar{z}} g \partial_z g^{-1}$$

is the Wess-Zumino-Witten action. In equation (17) $g_{(n)}$ denotes the operator representation on V_{Λ_n} of the element g of $G^{\mathbb{C}}$. The global implementation of this dependence requires

that the coupling constant k is an integer. Since any gauge field A can be rewritten as $A = g^{-1}\partial_{\bar{z}}g$ in terms of a complex gauge transformation g , all physical states are determined by its value at the trivial gauge field configuration $A_{\bar{z}} = 0$, i.e. $\Psi(0)$. On this configuration, the infinitesimal dependence of the physical states on the gauge field can be obtained [31] from Gauss' law (16)

$$\pi \frac{\delta}{\delta A_{\bar{z}}^a(z)} \Psi(A_{\bar{z}}) \Big|_{A=0} = \sum_{n=1}^N \frac{t_{(n)}^a}{z - z_n} \Psi(0) \quad (18)$$

and

$$\begin{aligned} \pi^2 \frac{\delta^2}{\delta A_{\bar{z}}^a(z) \delta A_{\bar{z}}^a(w)} \Psi(A_{\bar{z}}) \Big|_{A=0} &= \frac{k \dim G}{(z - w)^2} \Psi(0) + \sum_{n=1}^N \frac{t_{(n)}^a t_{(n)}^a}{(w - z_n)(z - z_n)} \Psi(0) \\ &+ \sum_{n \neq m} \frac{t_{(n)}^a t_{(m)}^a}{(w - z_n)(z - z_m)} \Psi(0), \end{aligned} \quad (19)$$

where the sum in the repeated index a is implicitly assumed and $1/(z - w)$ is the Green function of the operator $\frac{1}{\pi} \partial_{\bar{z}}$. From these expressions and the Sugawara form of the stress tensor ¹

$$T_{zz} = \frac{\pi^2}{2(k + h^\vee)} : \frac{\delta^2}{\delta A_{\bar{z}}^a(z) \delta A_{\bar{z}}^a(z)} : = \frac{\pi^2}{2(k + h^\vee)} \lim_{w \rightarrow z} \left[\frac{\delta^2}{\delta A_{\bar{z}}^a(z) \delta A_{\bar{z}}^a(w)} - \frac{k \dim G}{(z - w)^2} \right] \quad (20)$$

it is straightforward to show that the dependence on the translation of the insertion at z_n is governed by the Knizhnik-Zamolodchikov equation [28]

$$\partial_{z_n} \Psi(0) + \frac{1}{\kappa} \sum_{m \neq n} \frac{t_{(n)}^a t_{(m)}^a}{z_m - z_n} \Psi(0) = 0, \quad (21)$$

where $\kappa = k + h^\vee$ and h^\vee is the dual Coxeter number. In the same way the dilation transformation of the plane implies that

$$\sum_{n=1}^N z_n \partial_{z_n} \Psi(0) + \frac{1}{\kappa} H_G \Psi(0) = 0 \quad (22)$$

where the operator

$$H_G = -\frac{1}{2} \sum_{n \neq m} t_{(n)}^a t_{(m)}^a \quad (23)$$

in the second term of the equation is essentially the coupling term in the BCS Hamiltonian (eq (11)). This suggest that a possible slight modification of the Chern-Simons theory can trace back the geometric origin of the BCS Hamiltonian.

¹The metric dependence of the theory is induced both by the Bargmann normalization of quantum states (k) and the removal of spurious gauge degrees of freedom (h^\vee)

Let us consider the perturbation of Chern-Simons theory by a one-dimensional distribution of coloured charges. The effect of this perturbation is encoded by a new interaction term of the form

$$S_\xi(A) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dt \oint_{\mathcal{C}} dz \langle \xi, A_0(z, t) \rangle$$

where ξ is any element of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$, \mathcal{C} is a non-self-intersecting closed curve of \mathbb{C} which encloses all point-like insertions z_n of heavy coloured particles and where a one-dimensional ξ -coloured charge distribution is localized. Because of its ξ dependence this perturbation is not gauge invariant under gauge transformations not completely included in the normalizer of ξ . On the other hand, conformal invariance is broken by the special choice of the closed curve \mathcal{C} . However, the full Chern-Simons theory defined by $S(A, g) + S_\xi(A)$ can be quantized in a similar way and the corresponding quantum states are holomorphic functionals $\Psi(A_{\bar{z}})$ taking values in the same vector space $\otimes V_{\Lambda_n}$ because the new insertions have one-dimensional representations. The essential difference is encoded by the Gauss' law which now reads

$$\begin{aligned} D_{\bar{z}}^{ab} \frac{\delta}{\delta A_{\bar{z}}^b(z)} \Psi(A_{\bar{z}}) &= -\frac{k}{\pi} \partial_{\bar{z}} A_{\bar{z}}^a \Psi(A_{\bar{z}}) + \sum_{n=1}^N t_{(n)}^a \delta(z - z_n) \Psi(A_{\bar{z}}) \\ &\quad - \frac{1}{2\pi i} \oint_{\mathcal{C}} dz' \delta(z - z') \langle \xi, t^a \rangle \Psi(A_{\bar{z}}). \end{aligned} \quad (24)$$

This implies that the variation (19) now becomes

$$\begin{aligned} \pi^2 \frac{\delta^2}{\delta A_{\bar{z}}^a(z) \delta A_{\bar{z}}^a(w)} \Psi(A_{\bar{z}}) \Big|_{A=0} &= \frac{k \dim G}{(z - w)^2} \Psi(0) + \sum_{n=1}^N \frac{t_{(n)}^a t_{(n)}^a}{(w - z_n)(z - z_n)} \Psi(0) \\ &\quad + \sum_{n \neq m} \frac{t_{(n)}^a t_{(m)}^a}{(w - z_n)(z - z_m)} \Psi(0) - \sum_{n=1}^N \left(\frac{\xi_{(n)}}{z - z_n} + \frac{\xi_{(n)}}{w - z_n} \right) \Psi(0) + \langle \xi, \xi \rangle \Psi(0) \end{aligned} \quad (25)$$

where $\xi_{(n)}$ denotes the action of ξ on the representation Λ_n of the coloured particle sitting at z_n . Using the Sugawara form of the stress tensor (20) it is easy to show that the physical states must satisfy a modified Knizhnik-Zamolodchikov equation [28]

$$\partial_{z_n} \Psi(0) + \frac{1}{\kappa} \xi_{(n)} \Psi(0) + \frac{1}{\kappa} \sum_{m \neq n} \frac{t_{(n)}^a t_{(m)}^a}{z_m - z_n} \Psi(0) = 0. \quad (26)$$

In a similar way the dilation transformation now yields the equation

$$\sum_{n=1}^N z_n \partial_{z_n} \Psi(0) + \frac{1}{\kappa} H \Psi(0) = 0 \quad (27)$$

and, what is remarkable, in the case $G = SU(2)$ the operator

$$H = -\frac{1}{2} \sum_{n \neq m} t_{(n)}^a t_{(m)}^a + \sum_{n=1}^N z_n \xi_{(n)} \quad (28)$$

does coincide up to a factor $1/g$ and additive constant with the reduced BCS Hamiltonian (11).

The solvability of the reduced BCS Hamiltonian worked out by Richardson can now be understood in this geometrical approach from the existence of a complete set of operators

$$R_n = \sum_{m \neq n} \frac{t_{(n)}^a t_{(m)}^a}{z_m - z_n} + \xi_{(n)} \quad (29)$$

commuting with H , $[R_n, H] = 0$, as first pointed out in Ref. [11]. The operators R_n also commute each other ($[R_n, R_m] = 0$) and provide a complete set of constants of motion. This remarkable feature generalizes the well known integrability of the Hamiltonian (23) which commutes with the Gaudin Hamiltonians. The only difference is due to the presence of the ξ term in (29). The commutativity of the Hamiltonians R_n also means that the connection (Knizhnik-Zamolodchikov connection) defined by them in the bundle of Chern-Simons states over the space of insertions \mathbb{R}^N is flat. In fact, this flatness property of the KZ connection is not only verified on the bundle of CS states but in a more general bundle of states where the extra condition of global colour invariance imposed by Gauss' law is not required. This colour neutrality condition comes from the integration of Gauss' law equation (24) and establishes that CS states can only take values on the subspace $V^0 \subset \otimes V_{\Lambda_n}$ of invariants under the diagonal action of the Cartan subgroup of G (here and below we take ξ in the Cartan subalgebra of G) and as we shall see later it will correspond to the half-filling regime of BCS superconductivity. In fact, this restriction can be ignored if we include an extra charge insertion z_0 at very large distance of all the other insertions with the only purpose of neutralizing their colour charge. In this way, the whole BCS theory emerges from CS in the singular limit $\kappa \rightarrow 0$ as we shall see in next section. This relation existing between Chern-Simons theory and the BCS theory can be even better understood from a elliptic point of view.

4 Knizhnik-Zamolodchikov-Bernard connection.

Let us consider now the Chern-Simons theory on a compact two dimensional space T^2 with genus one. This theory was studied in [32, 33]. Let us summarize the main results. To quantize the theory one has to pick up a complex structure in the torus $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$ with $\tau_2 \equiv \text{Im}\tau > 0$ and a local systems of coordinates z_n for the n^{th} Wilson line insertions. Then, quantization proceeds in a similar way as for the plane or the sphere. In Bargmann's quantization the states are holomorphic functionals of the $(0, 1)$ part of the gauge 1-form with values in the tensor product of the representation spaces $\otimes V_{\Lambda_n}$. The states are subject to the Gauss' law that once integrated tells us how the states transform under complex gauge transformation. Every (generic) gauge field can then be gauged out to a flat one of the form $A_u = \pi u d\bar{z}/\tau_2$ with $u \in \mathfrak{h}^{\mathbb{C}}$, the (complexified) Cartan subalgebra of G , and the states are effectively described in terms multidimensional theta functions $\theta(u)$ with values in the subspace $V^0 \subset \otimes V_{\Lambda_n}$. The states are also subject to extra regularity conditions. See [32, 33, 34] for details.

The dependence of the quantum states on the positions of the point-like insertions can be derived in a similar manner as in the planar case. However, in the elliptic case Gauss'

law (16) does not provides in a simple manner the infinitesimal variation of quantum states with respect to arbitrary gauge fields. In fact, the required Green function of the operator $D_{\bar{z}} = \partial_{\bar{z}} + \pi u/\tau_2$ does depend on u . The trick is to remove the u dependence from the differential operator $D_{\bar{z}} = \partial_{\bar{z}} + \pi u/\tau_2$ by means of a singular gauge transformation

$$\Psi(u) = e^{\pi k \langle u, u \rangle / 2\tau_2} \prod_{n=1}^N e^{-\pi u_{(n)} (\bar{z}_n - z_n) / \tau_2} \gamma(u).$$

The u dependence is traded in this manner by a non-trivial twisted boundary condition for Chern-Simons states and the Green functions of the operator $\partial_{\bar{z}}$. We further extract a normalization factor accounting for the volume of the complex gauge orbits

$$\Pi(u) = e^{\pi i \dim G / 12} \prod_{\alpha > 0} \left(e^{\pi i \langle u, \alpha \rangle} - e^{-\pi i \langle u, \alpha \rangle} \right) \prod_{l=1}^{\infty} \left(1 - e^{2\pi i \tau l} \right)^{\text{rank} G} \prod_{\alpha} \left(1 - e^{2\pi i \tau l + 2\pi i \langle u, \alpha \rangle} \right)$$

with the product in α (resp. $\alpha > 0$) extending to the set of roots (resp. positive roots) of G . If we then define θ by

$$\gamma(u) = \Pi(u) \theta(u),$$

the dependence of the CS quantum states on the coordinates of the point-like charged insertions and on the elliptic modular parameter τ [35] can be expressed in a simple geometrical meaningful form: Chern-Simons states describe horizontal sections

$$\nabla_{\mathcal{A}} \theta(u) = 0$$

with respect to the flat Knizhnik-Zamolodchikov-Bernard (KZB) connection \mathcal{A} defined by

$$\mathcal{A}_{\bar{\tau}} = 0, \quad \mathcal{A}_{\bar{z}_n} = 0, \quad \mathcal{A}_{\tau} = \frac{1}{\kappa} H_0(\tau, \underline{z}), \quad \mathcal{A}_{z_n} = \frac{1}{\kappa} H_n(\tau, \underline{z}). \quad (30)$$

with

$$H_0(\tau, \underline{z}) = \frac{i}{4\pi} \Delta_u + \frac{i}{8\pi} \sum_{n,m=1}^N \left(-2 \sum_{\alpha} \partial_x \sigma_{\langle u, \alpha \rangle} (z_n - z_m) (e_{\alpha})_{(n)} (e_{-\alpha})_{(m)} + \sum_{j=1}^r (\rho(z_n - z_m)^2 + \rho'(z_n - z_m)) h_{(n)}^j h_{(m)}^j \right), \quad (31)$$

$$H_n(\tau, \underline{z}) = \sum_{j=1}^r h_{(n)}^j \partial_{u^j} - \sum_{m \neq n} \left(\sum_{\alpha} \sigma_{\langle u, \alpha \rangle} (z_n - z_m) (e_{\alpha})_{(n)} (e_{-\alpha})_{(m)} + \sum_{j=1}^r \rho(z_n - z_m) h_{(n)}^j h_{(m)}^j \right) \quad (32)$$

on the vector bundle of Chern-Simons states over the moduli of punctured elliptic curves. In equations (31)(32) ρ and σ_x denote the Green functions of $\frac{1}{\pi} \partial_{\bar{z}}$ that can be expressed in terms of the Jacobi theta function

$$\vartheta_1(z) = \sum_l (-1)^l e^{\pi i (l+1/2)^2 \tau + 2\pi i z (l+1/2)}$$

by

$$\rho(z) = \frac{\vartheta'_1(z)}{\vartheta_1(z)}$$

and

$$\sigma_x(z) = \frac{\vartheta'(0)\vartheta(x-z)}{\vartheta(x)\vartheta(z)},$$

and we use the notation of Refs. [36, 37, 38]: e_α represents the step generator associated to root α of \mathfrak{g} and $\{h_i, i = 1, \dots, r\}$ is an orthonormal basis of the Cartan subalgebra, so that $u = \sum_j u_j h_j$.

A natural hermitian structure in this bundle was introduced in Ref. [39] by means of the scalar product à la Bargmann, in that paper it was also shown that KZB connection is the unique $(1,0)$ unitary connection in this bundle.

It is easy to check that the KZB connection leave V^0 invariant and restricted to V^0 is flat for any value of κ . This implies, in particular, that Hamiltonians H_n restricted to V^0 commute for $n = 0, 1, \dots, N$ and determine then a quantum integrable system that admits a family of eigenvectors using the appropriate Bethe ansatz [39]. Below we shall discuss the solutions to the problem of eigenvalues.

Notice that in this case the equation corresponding to a global dilation is more involved than in planar case which makes the connection with a Hamiltonian of BCS type less transparent. However, we shall show that in the limit $\tau \rightarrow i\infty$ where the torus degenerates into the cylinder we shall recover a connection with the BCS theory. In particular, after rescaling $\tau \mapsto \eta\tau$ and taking the limit $\eta \rightarrow \infty$ we obtain the new family of commuting Hamiltonians, \tilde{H}_0 and \tilde{H}_n that read:

$$\tilde{H}_0(\underline{z}) = \frac{i}{4\pi}\Delta_u - \frac{\pi i}{4} \sum_{\alpha} \frac{\mathbf{e}_{\alpha}\mathbf{e}_{-\alpha}}{\sin^2(\pi\langle u, \alpha \rangle)}, \quad (33)$$

$$\begin{aligned} \tilde{H}_n(\underline{z}) = & \sum_{j=1}^r h_{(n)}^j \partial_{u^j} - \pi \sum_{m \neq n} \cot(\pi(z_n - z_m)) t_{(n)}^a t_{(m)}^a \\ & + \pi \sum_{m \neq n} \sum_{\alpha} \cot(\pi\langle u, \alpha \rangle) (e_{\alpha})_{(n)} (e_{-\alpha})_{(m)} \end{aligned} \quad (34)$$

with $\mathbf{e}_{\alpha} = \sum_n (e_{\alpha})_{(n)}$.

Next we take the limit from the cylinder to the plane by sending variables z_n and u to the origin, we rescale $z_n \mapsto \zeta z_n$ and $u \mapsto \zeta u$, $\tilde{R}_0 = \zeta^2 \tilde{H}_0$, $\tilde{R}_n = \zeta \tilde{H}_n$. In the limit of $\zeta \rightarrow 0$ the new commuting Hamiltonians are:

$$\tilde{R}_0(\underline{z}) = \frac{i}{4\pi}\Delta_u - \frac{i}{4\pi} \sum_{\alpha} \frac{\mathbf{e}_{\alpha}\mathbf{e}_{-\alpha}}{\langle u, \alpha \rangle^2}, \quad (35,a)$$

$$\tilde{R}_n(\underline{z}) = \sum_{m \neq n} \frac{t_{(m)}^a t_{(n)}^a}{z_m - z_n} + \sum_{j=1}^r h_{(n)}^j \partial_{u^j} - \sum_{m \neq n} \sum_{\alpha} \frac{(e_{\alpha})_{(m)} (e_{-\alpha})_{(n)}}{\langle u, \alpha \rangle} \quad (35,b)$$

We finally look for solutions to the eigenvalue problem for the Hamiltonians (35) of the form

$$\Upsilon(u, \underline{z}) = e^{\langle \xi, u \rangle} F(u, \underline{z}) \quad (36)$$

such that $\xi \in \mathfrak{h}^\mathbb{C}$ and $F(u/\epsilon, \underline{z})$ is for generic u and \underline{z} holomorphic in $\epsilon \in \mathbb{C}$, i. e. we assume

$$F(u/\epsilon, \underline{z}) = F_0(\underline{z}) + \epsilon F_1(u, \underline{z}) + \dots \quad (37)$$

If solutions of this form exist (later we shall see that this is the case) then F_0 is an eigenvector of Hamiltonians, $R_n = \lim_{u \rightarrow \infty} e^{-\langle \xi, u \rangle} \tilde{R}_n e^{\langle \xi, u \rangle}$, and

$$R_n = \sum_{m \neq n} \frac{t_{(m)}^a t_{(n)}^a}{z_m - z_n} + \xi_{(n)} \quad (38)$$

In the case of $SU(2)$ these are the conserved quantities build by CRS [11]. The reduced BCS Hamiltonian is essentially given by $H = \sum_n z_n R_n$ (28) and hence it obviously commutes with all R_n .

5 Bethe ansatz and horizontal sections.

We shall address now the question of finding solutions to the equations that appear in the different limits. Following [38] we define for $\beta \in \mathfrak{h}^*$

$$\omega_\beta(z) = \sigma_{\langle u, \beta \rangle}(z) dz + \frac{i}{2\pi} \partial_x \sigma_{\langle u, \beta \rangle}(z) d\tau$$

note that ω is a closed form in z, τ space.

Now decompose the sum of highest weights in simple roots (we assume that the decomposition exists):

$$\sum_{s=1}^K \alpha_s = \sum_{n=1}^N \Lambda_n$$

and for any partition $\underline{K} = (K_1, \dots, K_N)$ of K in N non-negative integers, relabel the roots $\underline{\alpha} = (\alpha_1, \dots, \alpha_K) \equiv (\alpha_{1,1}, \dots, \alpha_{1,K_1}, \alpha_{2,1}, \dots, \alpha_{N,K_N})$. Introduce a complex variable y_s for any simple root α_s and denote $\underline{y} = (y_1, \dots, y_K) \equiv (y_{1,1}, \dots, y_{1,K_1}, y_{2,1}, \dots, y_{N,K_N})$.

Set the form

$$\begin{aligned} \Omega_{\underline{K}, \underline{\alpha}}(\tau, u, \underline{z}, \underline{y}) &= \omega_{\beta_{1,1}}(y_{1,1} - y_{1,2}) \wedge \omega_{\beta_{1,2}}(y_{1,2} - y_{1,3}) \wedge \dots \wedge \omega_{\beta_{1,K_1}}(y_{1,K_1} - z_1) \\ &\quad \wedge \dots \wedge \omega_{\beta_{N,1}}(y_{N,1} - y_{N,2}) \wedge \omega_{\beta_{N,2}}(y_{N,2} - y_{N,3}) \wedge \dots \wedge \omega_{\beta_{N,K_N}}(y_{N,K_N} - z_N) \\ &\quad \otimes_n (e_{-\alpha_{n,1}} \dots e_{-\alpha_{n,K_n}})_{(n)} |\Lambda\rangle \end{aligned} \quad (39)$$

where $\beta_{n,i} = \sum_{i'=1}^i \alpha_{n,i'}$ and $|\Lambda\rangle = \otimes_n |\Lambda_n\rangle$ stands for the tensor product of highest weight states of representations Λ_n .

And finally define

$$\Omega(\tau, u, \underline{z}, \underline{y}) = \sum_{\underline{K}} \sum_{\mathcal{P} \in S_K} (-1)^{|\mathcal{P}|} \Omega_{\underline{K}, \mathcal{P}}(\tau, u, \underline{z}, \mathcal{P} \underline{y})$$

The form Ω satisfies the equation

$$\partial S \wedge \Omega + d\tau \wedge H_0 \Omega + \sum_n dz_n \wedge H_n \Omega = 0 \quad (40)$$

where S is the multivalued holomorphic function

$$\begin{aligned} S(\tau, \underline{z}, \underline{y}) = & \sum_{n < n'} \langle \Lambda_n, \Lambda_{n'} \rangle \ln \tilde{\vartheta}_1(z_n - z_{n'}) - \sum_{n, s} \langle \Lambda_n, \alpha_s \rangle \ln \tilde{\vartheta}_1(z_n - y_s) \\ & + \sum_{s < s'} \langle \alpha_s, \alpha_{s'} \rangle \ln \tilde{\vartheta}_1(y_s - y_{s'}) \end{aligned} \quad (41)$$

with $\tilde{\vartheta}_1(z) = \vartheta_1(z)/\vartheta_1'(0)$, and ∂ stands for the holomorphic differential in τ, \underline{z} and \underline{y} variables.

Equation (40) was stated in [38] and plays the key role in order to get horizontal sections of the KZB connection (30) from integrals of form Ω in cycles with coefficients in a local system determined by $e^{\frac{1}{\kappa} S}$. See [38] for more details.

We are interested in a complementary aspect of eq. (40) that relates it directly to the Bethe ansatz solutions of Hamiltonians H_i . Let the function $G(\tau, u, \underline{z}, \underline{y})$ be given by

$$G d\tau \wedge d^N z \wedge d^K y = d\tau \wedge d^N z \wedge \Omega.$$

Then for any configuration \underline{y} that satisfies the Bethe equations

$$\partial_{y_s} S = 0, \quad s = 1, \dots, K$$

one derives from (40)

$$\begin{aligned} H_0 G &= -(\partial_\tau S) G \\ H_n G &= -(\partial_{z_n} S) G \end{aligned} \quad (42)$$

i.e. G is an eigenvector of H_0 and $H_n, n = 1, \dots, N$ with eigenvalues $-\partial_\tau S$ and $-\partial_{z_n} S$, respectively

Here we will need a generalization of these equations in order to get solutions with the asymptotic behavior of eq. (36), that will eventually lead to the Richardson equations. Following [39] take $\xi \in \mathfrak{h}^{\mathbb{C}}$ and define

$$\tilde{\Omega} = e^{\langle u, \xi \rangle} \Omega, \quad \tilde{S} = S + \frac{1}{4\pi i} |\xi|^2 \tau - \sum_n \langle \xi, \Lambda_n \rangle z_n + \sum_s \langle \xi, \alpha_s \rangle y_s, \quad (43)$$

from (40) one can derive a similar relation for the new objects

$$\partial \tilde{S} \wedge \tilde{\Omega} + d\tau \wedge H_0 \tilde{\Omega} + \sum_n dz_n \wedge H_n \tilde{\Omega} = 0 \quad (44)$$

and then one has eigenvectors

$$\begin{aligned} H_0 e^{\langle u, \xi \rangle} G &= -\left(\frac{1}{4\pi i} |\xi|^2 + \partial \tau S\right) e^{\langle u, \xi \rangle} G \\ H_n e^{\langle u, \xi \rangle} G &= (\langle \xi, \Lambda_n \rangle - \partial_{z_n} S) e^{\langle u, \xi \rangle} G \end{aligned} \quad (45)$$

provided \underline{y} is chosen so that

$$\sum_{s' \neq s} \langle \alpha_{s'}, \alpha_s \rangle \rho(y_{s'} - y_s) - \sum_n \langle \Lambda_n, \alpha_s \rangle \rho(z_n - y_s) = \langle \xi, \alpha_s \rangle$$

In order to obtain eigenvectors for the Hamiltonians of eqs. (35) we redefine $\xi \mapsto \xi/\zeta$ and take, as in the previous section, the two consecutive limits $\tau \rightarrow i\infty$ and $\underline{z}, u \rightarrow 0$, i.e. $\zeta \rightarrow 0$ in G.

After this limiting procedure $G \rightarrow F$ where

$$F(u, \underline{z}, \underline{y}) = \sum_{\underline{K}} \sum_{\mathcal{P} \in S_K} F_{\underline{K}, \mathcal{P} \underline{\alpha}} \quad (46)$$

with

$$\begin{aligned} F_{\underline{K}, \underline{\alpha}}(u, \underline{z}, \underline{y}) &= v_{\beta_{1,1}}(y_{1,1} - y_{1,2}) v_{\beta_{1,2}}(y_{1,2} - y_{1,3}) \cdots v_{\beta_{1,K_1}}(y_{1,K_1} - z_1) \\ &\quad \times \dots \times v_{\beta_{N,1}}(y_{N,1} - y_{N,2}) v_{\beta_{N,2}}(y_{N,2} - y_{N,3}) \cdots v_{\beta_{N,K_N}}(y_{N,K_N} - z_N) \\ &\quad \otimes_n (e_{-\alpha_{n,1}} \cdots e_{-\alpha_{n,K_n}})_{(n)} |\Lambda \rangle \end{aligned} \quad (47)$$

where

$$v_\beta(z) = \frac{1}{z} - \frac{1}{\langle u, \beta \rangle}$$

and the rest of the notation is as in eq. (39).

One then satisfies the eigenvalue equations:

$$\begin{aligned} \tilde{R}_0 e^{\langle u, \xi \rangle} F &= -\frac{1}{4\pi i} |\xi|^2 e^{\langle u, \xi \rangle} F \\ \tilde{R}_n e^{\langle u, \xi \rangle} F &= (\langle \xi, \Lambda_n \rangle - \partial_{z_n} T) e^{\langle u, \xi \rangle} F \end{aligned} \quad (48)$$

which hold for

$$\begin{aligned} T(\underline{z}, \underline{y}) &= \sum_{n < n'} \langle \Lambda_n, \Lambda_{n'} \rangle \ln(z_n - z_{n'}) - \sum_{n,s} \langle \Lambda_n, \alpha_s \rangle \ln(z_n - y_s) \\ &\quad + \sum_{s < s'} \langle \alpha_s, \alpha_{s'} \rangle \ln(y_s - y_{s'}) \end{aligned} \quad (49)$$

provided \underline{y} is chosen so that it fulfills the Bethe equations

$$\partial_{y_s} T + \langle \xi, \alpha_s \rangle = 0 \quad s = 1, \dots, K$$

or explicitly

$$\sum_{s' \neq s} \frac{\langle \alpha_{s'}, \alpha_s \rangle}{y_{s'} - y_s} - \sum_n \frac{\langle \Lambda_n, \alpha_s \rangle}{z_n - y_s} = \langle \xi, \alpha_s \rangle \quad s = 1, \dots, K. \quad (50)$$

Note that $F(u/\epsilon, \underline{z}, \underline{y})$ is for generic $\underline{z}, \underline{y}$ and u a polynomial in ϵ thus it admits the expansion of eq. (37) and

$$F_0(\underline{z}, \underline{y}) = \lim_{\epsilon \rightarrow 0} F(u/\epsilon, \underline{z}, \underline{y})$$

is an eigenvector of Hamiltonians R_n provided \underline{y} is a solution of (50), whose corresponding eigenvalue μ_n is given by $\langle \xi, \Lambda_n \rangle - \partial_{z_n} T$, i. e.

$$\mu_n = \langle \xi, \Lambda_n \rangle + \sum_{n' \neq n} \frac{\langle \Lambda_{n'}, \Lambda_n \rangle}{z_{n'} - z_n} - \sum_s \frac{\langle \alpha_s, \Lambda_n \rangle}{y_s - z_n} \quad (51)$$

In our construction we only considered the subspace V^0 of vectors in the zero total weight subspace, this was imposed from the very beginning, for only when restricted to this subspace the Hamiltonians H_n commute, defining an integrable system. However after the limiting procedure we arrive at Hamiltonians R_n and these have the property of commuting in the whole representation space. We should then produce solutions of arbitrary weight.

Suppose we search for an eigenvector of the Hamiltonians R_n of total weight λ_0 . In order to get it we introduce a new insertion at z_0 with representation of highest weight Λ_0 and vector space V_{Λ_0} . Now we decompose the sum of highest weights

$$\sum_{s=1}^L \alpha_s = \sum_{n=1}^N \Lambda_n + \Lambda_0$$

with Λ_0 and the last $L - K$ roots chosen to verify

$$\lambda_0 = \sum_{s=K+1}^L \alpha_s - \Lambda_0.$$

We reparametrize the corresponding screening variables

$$(y_1, \dots, y_L) \equiv (y_1, \dots, y_K, z_0 + \gamma_1, \dots, z_0 + \gamma_{L-K})$$

and send z_0 to ∞ while keeping y_s and γ_l finite. In the limit, the old Hamiltonians R_n decouple from the new insertion, the Bethe equations (50) for $s = 1, \dots, K$ are unchanged (the new terms coming from the new insertion and screening variables vanish in the limit) and the eigenvectors factorize in $F_0 \otimes f$ where $f \in V_{\Lambda_0}$ has weight $-\lambda_0$. As the total weight is zero F_0 is an eigenvector of Hamiltonians R_n (provided the Bethe equations are fulfilled) of weight λ_0 . In practice we can forget about the new insertion and screening variables and simply solve the Bethe equations (50) and use the Bethe ansatz solution (46,47) but with a new set of simple roots so that

$$\sum_{s=1}^K \alpha_s = \sum_{n=1}^N \Lambda_n - \lambda_0, \quad (52)$$

and accordingly a different number of screening variables y_s .

This ends our main results which show the deep mathematical structure underlying the connection of BCS-like Richardson models with Chern-Simons theory.

6 Conclusions

In this paper we have shown how the BCS model, its exact solution, and integrability is related to the Chern-Simons theory. This has been done in two ways. First by adding a perturbation to the CS action on the plane by means of a one-dimensional coloured charge distribution, and second by taking an appropriated limit of the CS model on a torus with twisted boundary conditions for the WZW gauge fields. The construction is valid for any simple, simply connected compact Lie group G . The case of BCS corresponding to the choice $G = SU(2)$.

We find quite remarkable that the Chern-Simons theory not only gives effective descriptions of the fractional quantum Hall effect and the anyon superconductivity, but also of the standard BCS superconductivity. Though in the later case the formalism works, not in real space, but in the complex energy space. The space dimensionality enters only through the representations of the Lie group associated to every energy level, which for BCS is simply the level degeneracy.

This work adds one more member to the series: BCS/Gaudin models \rightarrow Integrable Vertex Models \rightarrow WZW models \rightarrow Chern-Simons theory, suggesting new research directions. One is the possible existence of integrable models associated to higher genus Riemann surfaces. This is an interesting but difficult problem, see [40] for a study of the genus 2 case.

Another question comes from the free field realization of the $SU(2)$ WZW model. This model has two screening operators J_{\pm} , where only one of them, i.e. J_+ , is required for the computation of correlators of the WZW primary fields with integer spins, while the second one is needed for primary fields with fractional spin [41, 42, 43]. In the limit where $k + 2 \rightarrow 0$, the positions of the screening operators $J_+(u)$ get frozen and satisfy the Richardson/Gaudin equations [12]. Thus J_+ can be regarded as a Cooper pair creation operator. The question is: what is the role, if any, of the second screening operator J_- and the associated primary fields with fractional spin? A conjecture is that they are associated to vortex operators in the BCS model since they produced additional zeros in the wave function. This possibility is currently under investigation.

Acknowledgements G.S. would like to thanks E.H. Kim and J. Dukelsky for discussions and the Department of Theoretical Physics of Zaragoza University for hospitality. We would like to thanks the Benasque Center of Physics where this work was completed. This work has been supported by the Spanish MCyT grants FPA2000-1252, BFM2000-1320-C02-01.

Appendix

The aim of this appendix is to give a more explicit expression of the basic equations (50,51) for their application to several problems in Condensed Matter and Nuclear Physics. To this end we shall exploit the structure of the simple Lie groups, together with their

representation theory [44].

The problem that is posed is the following: given a set of N complex variables $\{z_n\}_{n=1}^N$ together with a set of highest weights (h.w.) $\{\Lambda_n\}_{n=1}^N$ of a simple Lie group G and a generic element ξ belonging to the Cartan subalgebra $\mathfrak{h}^{\mathbb{C}}$ of G , find the solutions of “semiclassical Bethe equations” (50) for a set of K unknowns $\{y_s\}_{s=1}^K$, which are associated to a set of simple roots $\{\alpha_s\}_{s=1}^K$. Moreover the sets of h.w. and simple roots are related by the eq. (52) where λ_0 is the total weight of the eigenvector we look for. Once the solutions of (50) are found then eqs.(51) give the eigenvalues μ_n of the “perturbed” Gaudin operators R_n defined in eqs. (38).

A simple Lie Group G , with rank r , is characterized by r simple roots $\{\alpha_a\}_{a=1}^r$. It is convenient to describe the set $\{\alpha_s\}_{s=1}^K$ by the numbers $\{N_a\}_{a=1}^r$ of roots of the same type α_a . Then the label s amounts to the pair of labels (a, i) with $a = 1, \dots, r$ and $i = 1, \dots, N_a$, such that $K = \sum_{a=1}^r N_a$. Correspondingly we shall make the change of variables: $y_s = y_{(a,i)} \equiv E_i^a$. With these definitions eqs.(50) become

$$\sum_{b=1}^r \sum_{j=1}^{N_b} \frac{\langle \alpha_b, \alpha_a \rangle}{E_j^b - E_i^a} - \sum_{n=1}^N \frac{\langle \Lambda_n, \alpha_a \rangle}{z_n - E_i^a} = \langle \xi, \alpha_a \rangle \quad (53)$$

where $a = 1, \dots, r$ and $i = 1, \dots, N_a$. Of course the term $i = j, a = b$ must be excluded in (53). Similarly eqs. (51) read,

$$\mu_n = \langle \xi, \Lambda_n \rangle + \sum_{n' \neq n} \frac{\langle \Lambda_{n'}, \Lambda_n \rangle}{z_{n'} - z_n} - \sum_{a=1}^r \sum_{i=1}^{N_a} \frac{\langle \alpha_a, \Lambda_n \rangle}{E_i^a - z_n} \quad (54)$$

The various scalar products $\langle \cdot, \cdot \rangle$ appearing in these eqs. are well known in the theory of Lie algebras [44]. First of all, the scalar product of simple roots $\langle \alpha_a, \alpha_b \rangle$ is directly related to the Cartan matrix A_{ab} , which is defined as

$$A_{ab} = \langle \alpha_a, \alpha_b^{\vee} \rangle \quad (55)$$

where α_a^{\vee} is the coroot of α_a given by

$$\alpha_a^{\vee} = \frac{2\alpha_a}{|\alpha_a|^2} \quad (56)$$

For simply laced Lie groups (i.e. the ones in the ADE series) and in the normalization we have adopted, one has $|\alpha_a|^2 = 2, \forall a$, so that roots and coroots are equal, and the Cartan matrix is symmetric.

Next we have to compute the scalar products involving the h.w.’s Λ_n . Every weight λ is characterized by its expansion in the basis of the fundamental weight vectors $\{\omega_a\}_{a=1}^r$, i.e.

$$\lambda = \sum_{a=1}^r \lambda^a \omega_a \quad (57)$$

where $\{\lambda^a\}_{a=1}^r$ are integers called Dynkin labels (they are non negative for h.w.v). The basic property of the fundamental weights is

$$\langle \omega_a, \alpha_b^\vee \rangle = \delta_{ab} \quad (58)$$

Using eqs(56,57,58) one can deduce

$$\langle \lambda, \alpha_a \rangle = \frac{|\alpha_a|^2}{2} \lambda^a \quad (59)$$

The scalar product of two weights λ_1 and λ_2 is given by

$$\langle \lambda_1, \lambda_2 \rangle = \sum_{a,b=1}^r \lambda_1^a F_{ab} \lambda_2^b \equiv \lambda_1 \cdot F \cdot \lambda_2 \quad (60)$$

where F_{ab} is called the quadratic form matrix which is defined as

$$F_{ab} = \langle \omega_a, \omega_b \rangle = (A^{-1})_{a,b} \frac{|\alpha_b|^2}{2} \quad (61)$$

The last equality in eq.(61) follows from the relation between simple roots and fundamental weights,

$$\alpha_a = \sum_{b=1}^r A_{ab} \omega_b \quad (62)$$

The element $\xi \in \mathfrak{h}^{\mathbb{C}}$ will also be expanded in the basis of fundamental weights,

$$\xi = \sum_{a=1}^r \xi^a \omega_a \quad (63)$$

Then using eqs. (59) and (60) we get,

$$\begin{aligned} \langle \xi, \alpha_a \rangle &= \frac{|\alpha_a|^2}{2} \xi^a \\ \langle \xi, \lambda \rangle &= \xi \cdot F \cdot \lambda, \end{aligned} \quad (64)$$

Using the formulae introduced above we can finally write the Bethe eqs.(53) as

$$\sum_{b=1}^r \sum_{j=1}^{N_b} \frac{A_{ba}}{E_j^b - E_i^a} - \sum_{n=1}^N \frac{\Lambda_n^a}{z_n - E_i^a} = \xi^a \quad (65)$$

and the eigenvalues (54) as

$$\mu_n = \xi \cdot F \cdot \Lambda_n + \sum_{n' \neq n} \frac{\Lambda_{n'} \cdot F \cdot \Lambda_n}{z_{n'} - z_n} - \sum_{a=1}^r \sum_{i=1}^{N_a} \frac{\Lambda_n^a |\alpha_a|^2 / 2}{E_i^a - z_n} \quad (66)$$

where Λ_n^a are the Dynkin labels of the h.w. Λ_n . In reference [44] can be found the explicit expressions of the Cartan and quadratic form matrices, so the remaining data to be fixed

are the representations associated to the sites as well as the weight λ_0 (see below for examples).

Example 1: $G = SU(2)$

$SU(2)$ is the unique simple Lie group with rank 1 (i.e. $r = 1$), with a simple root $\alpha_1 = \sqrt{2}$ and a fundamental weight $\omega_1 = \alpha_1/2 = 1/\sqrt{2}$. The Cartan and the quadratic form “matrices” are given by $A_{11} = 2$ and $F_{11} = 1/2$. The Dynkin label of a h.w. Λ is twice the spin, i.e. $\Lambda^1 = 2s$, ($s = 0, 1/2, 1, \dots$). With the notation $N_1 = M$ (M being the number of Cooper pairs) and $\xi^1 = -1/g$, (g being the BCS coupling constant), eqs.(65) yield the Richardson eqs. (6)

$$\sum_{j \neq i}^M \frac{2}{E_j - E_i} - \sum_{n=1}^N \frac{2s_n}{z_n - E_i} + \frac{1}{g} = 0 \quad (67)$$

associated to the BCS Hamiltonian, while eqs.(66) yield the eigenvalues of the CRS conserved quantities [11, 12, 13].

$$\mu_n = -\frac{s_n}{g} + \sum_{n' \neq n} \frac{2s_{n'}s_n}{z_{n'} - z_n} - \sum_{i=1}^M \frac{2s_n}{E_i - z_n} \quad (68)$$

In the BCS case, N is the number of energy levels which are occupied by electrons pairs, z_n is twice the single particle energy (i.e. $z_n = 2\epsilon_n$), $2s_n$ is the maximal number of pairs that can occupy that level.

Denoting by s_0 the spin of the weight λ_0 appearing in eq.(52) one gets

$$s_0 = \sum_{n=1}^N s_n - M \quad (69)$$

In the case where $s_n = 1/2$, $\forall n$, the half-filling constraint (i.e. as many electrons as energy levels) is given by $N = 2M$, which from (69) corresponds to $s_0 = 0$.

Before we consider more general groups, we shall discuss in more detail eq.(52) regarding the relation between the number of roots N_a and the weights Λ_n and λ_0 . First of all this eq. can be written as

$$\sum_{a=1}^r N_a \alpha_a = \sum_{n=1}^N \Lambda_n - \lambda_0 \quad (70)$$

The problem is to find the numbers N_a given N , $\{\Lambda_n\}_{n=1}^N$ and λ_0 . Actually there may not exist a solution of this eq. with all N_a non negative integers.

A particular case where we know there must exist a solution is for all the Λ'_n s the adjoint irrep and $\lambda_0 = 0$. The h.w. of the adjoint representation is given by the highest root θ

$$\theta = \sum_{a=1}^r d_a \alpha_a \quad (71)$$

where d_a are positive integers called the marks (we use a different notation from reference [44]). For $SU(n)$ groups $d_a = 1, \forall a$. Setting $\Lambda_n = \theta, \forall n$ and $\lambda_0 = 0$ in (70) the solution is given by

$$N_a = d_a N \quad (72)$$

Let us now consider the more general case. Using eq.(62) in (70) we get

$$\sum_{b=1}^r N_b A_{ba} = \sum_{n=1}^N \Lambda_n^a - \lambda_0^a \quad (73)$$

Multiplying by the inverse of the Cartan matrix and using eq.(61) we finally get the desired expression:

$$N_a = \frac{2}{|\alpha_a|^2} \sum_{b=1}^r F_{ba} \left(\sum_n \Lambda_n^b - \lambda_0^b \right) \quad (74)$$

which obviously may not yield integers values for N_a .

Example 2: $G = SU(r+1)$

The Lie group $A_r \equiv SU(r+1)$ has rank r . We shall choose N insertions in the fundamental representation, whose Dynkin labels are given by $\Lambda = (1, 0, \dots, 0)$. From eq.(73) we derive then

$$\begin{aligned} \lambda_0^1 &= N - 2N_1 + N_2 \\ \lambda_0^2 &= -2N_2 + N_1 + N_3 \\ &\vdots \\ \lambda_0^r &= -2N_r + N_{r-1}. \end{aligned} \quad (75)$$

If we choose $\lambda_0 = 0$ then the number of roots N_a follows from eq.(74)

$$N_a = N \frac{r+1-a}{r+1} \quad (76)$$

Hence N must be a multiple of $r+1$. This reflects the fact that the tensor product of $r+1$ copies of the fundamental irrep contains the identity.

The Bethe eqs. in the case of $SU(3)$ and arbitrary irreps is given by

$$\begin{aligned} \sum_{j=1}^{N_1} \frac{2}{E_j^1 - E_i^1} - \sum_{j=1}^{N_2} \frac{1}{E_j^2 - E_i^1} - \sum_{n=1}^N \frac{\Lambda_n^1}{z_n - E_i^1} &= \xi^1 \\ \sum_{j=1}^{N_2} \frac{2}{E_j^2 - E_i^2} - \sum_{j=1}^{N_1} \frac{1}{E_j^1 - E_i^2} - \sum_{n=1}^N \frac{\Lambda_n^2}{z_n - E_i^2} &= \xi^2 \end{aligned} \quad (77)$$

Other representations and groups can be worked out similarly.

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